$N = 4$ BPS black holes and octonionic twistors

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ABSTRACT: Stationary, spherically symmetric solutions of $\mathcal{N} = 2$ supergravity in 3+1 dimensions have been shown to correspond to holomorphic curves on the twistor space of the quaternionic-Kähler space which arises in the dimensional reduction along the time direction. In this note, we generalize this result to the case of 1/4-BPS black holes in $\mathcal{N} = 4$ supergravity, and show that they too can be lifted to holomorphic curves on a "twistor space" Z, obtained by fibering the Grassmannian $F = \mathrm{SO}(8)/\mathrm{U}(4)$ over the moduli space in three-dimensions $\mathrm{SO}(8, n_v + 2)/\mathrm{SO}(8) \times \mathrm{SO}(n_v + 2)$. This provides a kind of octonionic generalization of the standard constructions in quaternionic geometry, and may be useful for generalizing the known BPS black hole solutions, and finding new non-BPS extremal solutions.

KEYWORDS: Black Holes in String Theory, Black Holes, String Duality.

Contents

1. Introduction

While supersymmetry often leads to solvability, its full power reveals itself only when translated into holomorphy. For supergravity theories with $\mathcal{N} = 2$ supersymmetries in 4 dimensions, this may be achieved using projective superspace [1] or harmonic superspace techniques [2]. From a mathematical viewpoint, these techniques are closely related to twistors, whose purpose is to enforce holomorphy in all complex structures at once. While these methods have often been used to restrict the possible terms in the low energy effective action, they can also be useful in constructing actual supersymmetric solutions of the field equations [3–6].

In particular, in [3, 5, 7], it was shown that spherically symmetric BPS black hole solutions in $\mathcal{N} = 2$ supergravity correspond to holomorphic curves in Z, the twistor space of the quaternionic-Kähler moduli space $\mathcal{M}_3$ which appears after dimensional reduction along the time direction. This translation of supersymmetry to holomorphy was then used to recover the known spherically symmetric BPS solutions, and to obtain the exact quantum wave function for the radial evolution of the scalar fields, at two derivative order.

It is likely that multi-centered BPS solutions could also be understood or generalized using the same geometrical framework.

The purpose of this note is to extend the techniques of [3, 5] to the case of $\mathcal{N} = 4$ supergravity with n_v vector multiplets in 3+1 dimensions. Since the moduli space for such theories in three-dimensions is a symmetric space $\mathcal{M}_3 = K_3 \backslash G_3 = \mathrm{SO}(8) \times \mathrm{SO}(n_v + 2)] \backslash \mathrm{SO}(8, n_v + 2)$ [8, 9], spherically symmetric solutions can be readily obtained by exponentiating a one-parameter subgroup and so hold little mystery. Nevertheless, with a view to a possible extension to the multi-centered case, or to the inclusion of higher derivative corrections such as the one uncovered in [10], it is interesting to see how the translation of supersymmetry to holomorphy takes place.

While an approach based on harmonic superspace ideas is also possible [11], we prefer to follow the road of projective superspace, and the guidance of 1/4-BPS black holes. By including the pair of Killing spinors preserved by the solution into the phase space of the dynamical system governing the radial evolution equations, we show that BPS solutions can again be lifted to holomorphic curves in the "twistor space" $Z = M_3 \backslash G_3 = [\mathrm{U}(4) \times \mathrm{SO}(n_v + 2)] \backslash \mathrm{SO}(8, n_v + 2)$, whose fiber F over any point in $\mathcal{M}_3$ is the Grassmanniann $\mathrm{U}(4) \backslash \mathrm{SO}(8) = [\mathrm{SO}(2) \times \mathrm{SO}(6)] \backslash \mathrm{SO}(8)$. The twistor space Z appears in Bryant's classification of twistor spaces of symmetric spaces [12], and its relevance for black holes was first suggested in [5]. In contrast to the standard twistor space for quaternionic-Kähler manifolds, Z does not have a (twisted) holomorphic contact form, but instead an antisymmetric 4×4 matrix of them, transforming into each other under the local $\mathrm{SU}(4)$ action. This complication prevents us from constructing a complex coordinate system adapted to the Heisenberg group of symmetries which is crucial for applications to black holes, although there is little doubt that such a system exists. Similarly, we fail to produce the most general black hole wave function, but we do exhibit some holomorphic wave functions.

The outline of this note is as follows. In section 2, we review the equivalence between stationary, spherically symmetric solutions in 4D and geodesic motion in 3D, derive the supersymmetry conditions, and obtain BPS and non BPS solutions by exponentiating one-parameter subgroups in G_3. In section 3, we construct the twistor space Z, first in a "bottom-up" approach suggested by the black hole problem, and second in a more algebraic "top-down" approach analogous to the construction in [13]. The equivalence between BPS solutions and holomorphic curves is explained in section 3.4. In the appendices, we state our conventions for $\mathrm{SO}(8)$ Dirac matrices, and review some general facts about nilpotent co-adjoint orbits in orthogonal groups.

2. Black holes and geodesics

2.1 $\mathcal{N} = 4$ supergravity in four dimensions

Consider $\mathcal{N} = 4$ supergravity in 3+1 dimensions with n_v vector multiplets [14, 15]. The spectrum consists of the graviton, 4 gravitini, $n_v + 6$ Abelian vector fields, $n_v + 1$ Majorana spinors and $6n_v + 2$ real scalar fields parametrizing the moduli space

$$\mathcal{M}_4 = \frac{Sl(2, \mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6, n_v, \mathbb{R})}{\mathrm{SO}(6) \times \mathrm{SO}(n_v)}, \tag{2.1}$$

The first factor in (2.1) corresponds to the axion-dilaton field $\tau = \tau_1 + i\tau_2$ from the gravity supermultiplet, while the second factor corresponds to the scalars in the n_v vector multiplets. The U(1) and SO(6) subgroups in the denominator of (2.1) correspond to the R-symmetry group U(4).

An $\mathcal{N} = 4$ supergravity with $n_v = 22$ vector multiplets is known to arise by toroidal compactification of the heterotic string on T^6. Theories with fewer vector multiplets can be constructed by freely acting orbifolds of this model [16]. In these cases, as long as $n_v \geq 6$, it is convenient to parametrize the second factor of (2.1) by the coset element

$$e_{6,n_v} = \begin{pmatrix} e_6 & 0 & 0 \\ 0 & \mathbb{I}_{n_v-6} & 0 \\ 0 & 0 & e_6^{-T} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_6 & W & B - \frac{1}{2}W^T\eta_{6,n_v}W \\ 0 & \mathbb{I}_{n_v-6} & -W^T\eta_{6,n_v} \\ 0 & 0 & \mathbb{I}_6 \end{pmatrix} \in \mathrm{SO}(6,n_v,\mathbb{R}) \qquad (2.2)$$

which preserves the signature $(+_6, -_{n_v})$ metric

$$\eta_{6,n_v} = \begin{pmatrix} & & \mathbb{I}_6 \\ & -\mathbb{I}_{n_v-6} & \\ \mathbb{I}_6 & & \end{pmatrix} \qquad (2.3)$$

Here, $e_6 \in \frac{GL(6,\mathbb{R})}{SO(6)}$ is the viel-bein for the metric on T^6, which can be chosen in upper triangular form, B is an antisymmetric 6×6 matrix corresponding to the Kalb-Ramond two-form pulled back to T^6, and W is a $6 \times (n_v - 6)$ matrix corresponding to the Wilson lines of the n_v Abelian gauge fields in the Cartan subgroup of the 10D gauge group (or its projection in the case of CHL compactifications). When $n_v < 6$, one may instead use the decomposition of $\mathfrak{so}(6,n_v,\mathbb{R})$ as the sum of a compact (i.e. antisymmetric) and a non-compact (symmetric) element, and parametrize the second factor in (2.1) by a real $6 \times n_v$-matrix A,

$$e_{6,n_v} = \exp\begin{pmatrix} 0_6 & A \\ A^T & 0_{n_v} \end{pmatrix}, \quad \eta_{6,n_v} = \begin{pmatrix} \mathbb{I}_6 & \\ & -\mathbb{I}_{n_v} \end{pmatrix}. \qquad (2.4)$$

For type II compactifications on $K_3 \times T^2$, or freely-acting orbifolds thereof, other parametrizations adapted to the SO(4, 20) mirror symmetry group of K_3 are more convenient (see e.g. [17]).

Irrespective of the choice of coset representative, the invariant metric on the second factor in (2.1) can be obtained by decomposing the right-invariant one-form $\theta_{6,n_v} = de_{6,n_v} \cdot e_{6,n_v}^{-1}$ into a sum $h_{6,n_v} + p_{6,n_v}$ of its compact and non-compact parts, and forming a quadratic combination of the non-compact part p_{6,n_v} which is invariant under the action of the maximal compact subgroup $\mathrm{SO}(6) \times \mathrm{SO}(n_v)$. Combining it with the standard line element on the upper-half plane, the moduli space metric is thus given by

$$ds^2_{\mathcal{M}_4} = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} + \mathrm{Tr}(p_{6,n_v}^2) = \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} - \frac{1}{4}\mathrm{Tr}(dM \cdot dM^{-1}) \qquad (2.5)$$

where $M \equiv e_{6,n_v}^T \cdot e_{6,n_v}$ is a symmetric matrix in $\mathrm{SO}(6,n_v,\mathbb{R})$, invariant. Under the action of an element $g \in \mathrm{SO}(6,n_v)$, e_{6,n_v} transforms by right-multiplication by g followed by a

compensating left-multiplication by an element in $\mathrm{SO}(6) \times \mathrm{SO}(n_v)$ so as to restore the gauge choice (2.2), while M transforms linearly in the symmetric representation $M \to g^T M g$.

Including the $n_v + 6$-dimensional gauge fields $A_{\mu\nu}^\Lambda$ ($\Lambda = 1 \dots n_v + 6$), arranged as a vector of $\mathrm{SO}(6, n_v)$, the complete bosonic action of $\mathcal{N} = 4$ supergravity at two-derivative level is given by

$$
\begin{aligned}
S_4 = \int d^4 x \sqrt{-g} \bigg[& R - \frac{1}{2} \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} + \frac{1}{8} \mathrm{Tr}(dM \cdot dM^{-1}) \\
& - \frac{1}{4} \tau_2 F_{\mu\nu}^T \cdot M^{-1} \cdot F^{\mu\nu} + \frac{1}{4} \tau_1 F_{\mu\nu}^T \cdot \eta_{6,n_v} \cdot \tilde{F}^{\mu\nu} \bigg] .
\end{aligned}
\tag{2.6}
$$

While the action is manifestly invariant under $\mathrm{SO}(6, n_v, \mathbb{R})$, the $Sl(2, \mathbb{R})$ symmetry is only visible at the level of the equations of motion. According to string duality conjectures, the quantum theory is invariant under an arithmetic subgroup of $Sl(2, \mathbb{R}) \times \mathrm{SO}(6, n_v, \mathbb{R})$, whose precise definition depends on the model under consideration.

2.2 Reduction to 3D

In order to study stationary solutions, with metric

$$
ds_4^2 = -e^{2U}(dt + \omega)^2 + e^{-2U} ds_3^2 ,
\tag{2.7}
$$

it is convenient to reduce the 4D $\mathcal{N} = 4$ supergravity theory along the time direction to a $\mathcal{N} = 8$ theory supergravity in three Euclidean dimensions [5, 18–21, 7]. After dualizing one-forms into pseudo-scalars, all bosonic degrees of freedom can be described by a non-linear sigma model with non-Riemannian target space

$$
\mathcal{M}_3^* = G_3 / K_3^* = \frac{\mathrm{SO}(8, n_v + 2, \mathbb{R})}{\mathrm{SO}(6, 2) \times \mathrm{SO}(2, n_v)} ,
\tag{2.8}
$$

coupled to 3D Euclidean gravity. The moduli space (2.8) is related to the more familiar Riemannian space arising in the reduction along a space-like direction [8, 9]

$$
\mathcal{M}_3 = G_3 / K_3 = \frac{\mathrm{SO}(8, n_v + 2, \mathbb{R})}{\mathrm{SO}(8) \times \mathrm{SO}(n_v + 2)}
\tag{2.9}
$$

by analytic continuation, as we describe presently. As in [22], it is convenient to parametrize $\mathcal{M}_3$ by choosing a metric

$$
\eta_{8, n_v+2} = \begin{pmatrix} & & \mathbb{I}_2 \\ & \eta_{6,n_v} & \\ \mathbb{I}_2 & & \end{pmatrix}
\tag{2.10}
$$

and a coset representative in (partial) Iwasawa gauge,

$$
e_{8,n_v+2} = \begin{pmatrix} \frac{e^{-U}}{\sqrt{\tau_2}} & \frac{e^{-U}\tau_1}{\sqrt{\tau_2}} & 0 & 0 \\ 0 & e^{-U}\sqrt{\tau_2} & & \\ & 0 & e_{6,n_v} & 0 \\ & & e^U\sqrt{\tau_2} & 0 \\ 0 & 0 & -\frac{e^U \tau_1}{\sqrt{\tau_2}} & \frac{e^U}{\sqrt{\tau_2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \zeta^\Lambda & -\frac{1}{2}\zeta^\Lambda \zeta_\Lambda & \sigma - \frac{1}{2}\zeta^\Lambda \tilde{\zeta}_\Lambda \\ 0 & 1 & \tilde{\zeta}_\Lambda & -\sigma - \frac{1}{2}\zeta^\Lambda \tilde{\zeta}_\Lambda & -\frac{1}{2}\tilde{\zeta}_\Lambda \tilde{\zeta}^\Lambda \\ 0 & 0 & \mathbb{I}_{6,n_v} & -\zeta_\Lambda & -\tilde{\zeta}^\Lambda \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\tag{2.11}
$$

with $e_{6,n_v} \in \mathrm{SO}(6, n_v)/\mathrm{SO}(6) \times \mathrm{SO}(n_v)$ as in (2.2) or (2.4). The coordinates ζ^Λ and $\tilde\zeta_\Lambda$ correspond to the time-like component of the gauge fields $A_{\mu\nu}^\Lambda$ and their magnetic dual, while σ is the pseudo-scalar dual to the one-form ω. The indices on ζ^Λ and $\tilde\zeta_\Lambda$ are raised and lowered using the metric η_{6,n_v}. The decomposition (2.11) reflects the fact that under the subgroup $\mathbb{R}^+ \times Sl(2) \times \mathrm{SO}(6, n_v)$, $\mathrm{SO}(8, n_v + 2)$ admits the "real" 5-grading

$$1|_{-2} \oplus (2, n_v + 6)|_{-1} \oplus [(1,1) \oplus (3,1) \oplus +(1, so(6, n_v)]|_0 \oplus (2, n_v + 6)|_1 \oplus 1|_2 \qquad (2.12)$$

where the subscript indicates the charge under the $\mathbb{R}^+$ factor generated by the diagonal matrix $(\mathbb{I}_2, 0_6, -\mathbb{I}_2)$. The adjective "real" refers to the fact that each summand is invariant under the Cartan involution, so that the corresponding coordinates are real.

The invariant metric on (2.9) is obtained by the same prescription as above (2.5), namely by decomposing the right-invariant one-form

$$\theta_{8,n_v+2} = de_{8,n_v+2} \cdot e_{8,n_v+2}^{-1} \qquad (2.13)$$

into its compact and a non-compact parts, and taking the $\mathrm{SO}(8) \times \mathrm{SO}(n_v+2)$ invariant norm of the non-compact part. This is most easily done by changing basis such that the maximal compact subgroup corresponds to square blocks of size 8 and $n_v + 2$ on the diagonal,[1]

$$\eta_{8,n_v+2,K} = \begin{pmatrix} & \mathbb{I}_4 & \\ \mathbb{I}_4 & & \\ & & -\mathbb{I}_{n_v+2} \end{pmatrix} = \Omega_K^T\, \eta_{8,n_v+2}\, \Omega_K \qquad (2.14)$$

Such a change of basis is non-unique; a convenient choice is[2]

$$\Omega_K = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{I}_{n_v} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix} \qquad (2.15)$$

---

[1] The reason for choosing an off-diagonal metric for the SO(8) part will become apparent shortly.

[2] This choice ensures that the SU(3) subgroup of the 4D R-symmetry group SO(6) is mapped to a subgroup $\begin{pmatrix} 1 & & & \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{pmatrix}$ inside SU(4) $\subset$ SO(8).

In this new basis, the Cartan decomposition of θ_{8,n_v+2} is just the decomposition into blocks of dimension 8×8, $(n_v + 2) \times (n_v + 2)$ and $8 \times (n_v + 2)$:

$$\Omega_K^{-1} \, \theta_{8,n_v+2} \, \Omega_K = \begin{pmatrix} \theta_{AB} & p_{Aa} \\ p_{Aa} & \theta_{ab} \end{pmatrix} \tag{2.16}$$

where $A, B = 1 \ldots 8, a = 1 \ldots n_v + 2$. Conventionally, we take the non-compact part p_{Aa} to transform as a spinor of positive chirality under SO(8). The quadratic form $\begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix}$ appearing in (2.14) is recognized as the charge conjugation matrix C_{AB} in the spinor representation (see appendix A for our conventions for SO(8) spinors). The compact parts θ_{AB} and θ_{ab} correspond to the SO(8) and SO($n_v + 2$) spin connections, respectively. Thus, the right-invariant metric on $\mathcal{M}_3$ is given by

$$ds_{\mathcal{M}_3}^2 = p^{Aa} p^{Bb} C_{AB} \delta_{ab} \equiv g_{mn} d\phi^m d\phi^n \; . \tag{2.17}$$

The final result is

$$\begin{aligned} ds_{\mathcal{M}_3}^2 = dU^2 + ds_{\mathcal{M}_4}^2 + e^{-2U} \frac{(d\zeta^\Lambda + \tau d\tilde{\zeta}^\Lambda) \cdot M \cdot (d\zeta^\Lambda + \bar{\tau} d\tilde{\zeta}^\Lambda)}{\tau_2} \\ + e^{-4U}(d\sigma + \zeta^\Lambda d\tilde{\zeta}_\Lambda - \tilde{\zeta}_\Lambda d\zeta^\Lambda)^2 \end{aligned} \tag{2.18}$$

where $\tilde{\zeta}_\Lambda, \tilde{\zeta}^\Lambda, \sigma$ are identified as the time-component of the gauge field A_Λ and its magnetic dual $\tilde{A}^\Lambda$, and the NUT scalar dual to the connection one-form ω in (2.7). This relation between the moduli spaces in 3D and 4D is a straightforward generalization of the c-map encountered in the dimensional reduction of $\mathcal{N} = 2$ theories [23]. As mentioned above, the indefinite metric on the manifold $\mathcal{M}_3^*$ is obtained from (2.17) by analytically continuing $(\zeta^\Lambda, \tilde{\zeta}_\Lambda) \to -i(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$.

The supersymmetrization of the non-linear sigma model on $\mathcal{M}_3$ was studied in detail in [8, 24, 25]. We briefly summarize the main results following [25]. The $\mathcal{N} = 8$ super-symmetry algebra relies on the existence of seven almost complex hermitian structures f_n^{Pm} ($P = 2 \ldots 8$) satisfying the SO(7) Clifford algebra. From these, one may construct 28 two-forms $f^{\mu\nu} = f_{mn}^{\mu\nu} d\phi^m d\phi^n$ ($\mu = 1 \ldots 8$) via

$$f_{mn}^{PQ} = f_m^{[Pp} f_n^{Q]q} g_{pq} , \quad f_{mn}^{1P} = -f_{mn}^{P1} = g_{mp} f_n^{Pp} \; . \tag{2.19}$$

The tensors $f^{\mu\nu}$ are covariantly constant, and equal to the curvature of the SO(8) spin connection $Q^{\mu\nu} = Q_m^{\mu\nu} d\phi^m$,

$$dQ^{\mu\nu} + 2Q^{\rho[\mu} \wedge Q^{\nu]\rho} = \frac{1}{2} f^{\mu\nu} \; . \tag{2.20}$$

The fermionic degrees of freedom are most easily described by introducing a fermionic tensor $\chi^{m\mu}$ subject to the constraint

$$\chi^{m\mu} = \frac{1}{8} \left(\delta_\nu^\mu \delta_n^m - f_{\nu n}^{\mu m} \right) \chi^{n\nu} , \tag{2.21}$$

which projects down the number of components to $8(n_v+2)$. The supersymmetry variations of the gravitini ψ_M^μ ($M = 1, 2, 3$) and the dilatini $\chi^{m\mu}$, for vanishing fermionic background, are then written as

$$\delta\psi_M^\mu = D_M\epsilon^\mu, \quad \delta\chi^{m\mu} = \frac{1}{2}(\delta^{\mu\nu}\delta^{mn} - f_{mn}^{\mu\nu})\gamma^M\partial_M\phi^n\epsilon^\nu \tag{2.22}$$

For our purposes, it will be convenient to solve the constraint (2.21) explicitly, as

$$\chi^{m\mu} = e_{Aa}^m\Gamma_{AA'}^\mu\lambda^{aA'} \tag{2.23}$$

where $e_{Aa}^m = (p_m^{Aa})^{-1}$ is the inverse viel-bein afforded by the $\mathrm{SO}(8) \times \mathrm{SO}(n_v + 2)$ restricted holonomy, and $\Gamma_{AA'}^\mu$ are the $\mathrm{SO}(8)$ sigma matrices. In terms of the unconstrained spinor $\lambda^{aA'}$, the variation of the dilatini is given by

$$\delta\lambda_{A'}^a = p^{Aa}\,\Gamma_{A'A}^\mu\,\varepsilon_\mu\ . \tag{2.24}$$

Notice that the supersymmetry parameter ϵ^μ, dilatini $\lambda^{aA'}$ and bosonic derivatives p^{Aa} transform as the three inequivalent 8-dimensional representations of the R-symmetry group. Of course, one could use triality and permute the representations assigned to these objects.

2.3 Reduction to 1D

Upon further restricting to spherically symmetric solutions, with spatial metric

$$ds_3^2 = N^2(\rho)d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \tag{2.25}$$

the 3D non-linear sigma model reduces to the geodesic motion of a free particle on a real cone $\mathbb{R}^+ \times \mathcal{M}_3^*$ over (2.8), with action

$$S_1 = \int d\rho \left[\frac{N}{2} + \frac{1}{2N}\left(r'^2 - r^2 g_{mn}\phi'^m\phi'^n\right)\right]\ . \tag{2.26}$$

The equation of motion of N forces the Hamiltonian to vanish,

$$H_{\mathrm{WDW}} = (p_r^2) - \frac{1}{r^2}g^{mn}p_{\phi^m}p_{\phi^n} - 1 \equiv 0 \tag{2.27}$$

The system reduces to geodesic motion on $\mathcal{M}_3^*$, with momentum squared $g^{mn}p_{\phi^m}p_{\phi^n} = C^2$, and motion along r with conformally invariant Hamiltonian $(p_r^2) - C^2/r^2 - 1 = 0$. In particular, the phase space is given by the symplectic quotient of the cotangent bundle $T^*(\mathbb{R}^+ \times \mathcal{M}_3^*)$ by the first class constraint $H_{\mathrm{WDW}} = 0$. Extremal black holes necessarily have $C^2 = 0$ (although this condition is not sufficient), which gives a further first class constraint.

By the usual Noether procedure, Killing vectors $\kappa^m\partial_{\phi^m}$ of $\mathcal{M}_3^*$ yield conserved quantities $\kappa^m p_{\phi^m}$ for the geodesic motion on $\mathcal{M}_3^*$. Of particular interest are the isometries corresponding to shifts in the $\zeta, \tilde{\zeta}, \sigma$ directions,

$$P^\Lambda = \partial_{\tilde{\zeta}_\Lambda} - \zeta^\Lambda\partial_\sigma, \quad Q_\Lambda = -\partial_{\zeta^\Lambda} - \tilde{\zeta}_\Lambda\partial_\sigma, \quad K = \partial_\sigma \tag{2.28}$$

which satisfy the Heisenberg algebra

$$\left[P^\Lambda, Q_\Sigma\right] = -2\delta_\Sigma^\Lambda K\ . \tag{2.29}$$

Bona fide black holes have zero NUT charge $K = 0$, in which case P^Λ, Q_Σ correspond to the electric and magnetic charges of the black hole. In addition, the conserved quantity associated to the Killing vector

$$H = -\partial_U - \zeta^\Lambda \partial_{\zeta_\Lambda} - \tilde{\zeta}_\Lambda \partial_{\tilde{\zeta}_\Lambda} \tag{2.30}$$

is the ADM mass, provided one enforces the condition

$$U = \zeta^\Lambda = \tilde{\zeta}_\Lambda = \sigma = 0 \tag{2.31}$$

at spatial infinity.

While the conserved charges $P^\Lambda, Q_\Lambda, K, H$ appear universally in reductions of Einstein-Maxwell theories, in the present case there are additional conserved quantities due to the isometries of the scalar moduli space in 4 dimensions. For $n_v = 0$, the corresponding Killing vectors read

$$Y_0 = \tau_1 \partial_{\tau_1} + \tau_2 \partial_{\tau_2} + \frac{1}{2} \zeta^\Lambda \partial_{\zeta_\Lambda} - \frac{1}{2} \tilde{\zeta}_\Lambda \partial_{\tilde{\zeta}_\Lambda} \tag{2.32}$$

$$Y_+ = \partial_{\tau_1} - \tilde{\zeta}_\Lambda \partial_{\zeta^\Lambda} \tag{2.33}$$

$$Y_- = \frac{1}{2}(\tau_1^2 - \tau_2^2)\partial_{\tau_1} + \tau_1 \tau_2 \partial_{\tau_2} + \frac{1}{2} \zeta^\Lambda \partial_{\tilde{\zeta}_\Lambda} \tag{2.34}$$

and satisfy the $Sl(2,\mathbb{R})$ commutation relations

$$[Y_0, Y_\pm] = \pm Y_\pm, \quad [Y_+, Y_-] = Y_0. \tag{2.35}$$

In addition to the bosonic terms displayed in (2.26), the one-dimensional Lagrangian contains fermionic terms corresponding to the reduction of the $\mathcal{N} = 8$ supersymmetric sigma model in 3 dimensions along the sphere. This reduction was studied in detail in [5] in the $\mathcal{N} = 4$ case, and it was found that the reduction yields a one-dimensional sigma model with the same number of (spinorial) supersymmetries as in 3 dimensions.[3] Following the same analysis, we find that the conditions for radially symmetric solutions to preserve supersymmetry are given by

$$\exists \varepsilon_\mu \in \mathbb{C}^8 \backslash \{0\} \; / \; \forall a, A', \quad p^{Aa} \Gamma^\mu_{A'A} \varepsilon_\mu = 0 \quad \text{and} \quad r' = N. \tag{2.36}$$

The first condition implies that any linear combination of the $n_v + 2$ spinors p^{Aa} has zero norm. Put differently,

$$p^{Aa} p^{Bb} C_{AB} = 0. \tag{2.37}$$

This condition is in fact equivalent to the existence of ε_μ such that (2.36) is obeyed.[4] Clearly, it implies the extremality condition $C^2 = p^{Aa} p^{Bb} C_{AB} \delta_{ab} = 0$, but is considerably

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[3]To be precise, the supersymmetric completion of the 1D sigma model is known only in the sector involving $\mathcal{M}_3$ but not r and N.

[4]By an SO(8) rotation, the first spinor p^{A1} (rotated by the 8×8 upper-left block of Ω_K) can be chosen parallel to $(1, 0, 0, 0, i, 0, 0, 0)$; the second can be chosen to lie along $(0, 1, 0, 0, 0, i, 0, 0)$ up to the addition of the first, etc. In this basis, it is easy to check that all p^{Aa} are annihilated by $\Gamma^\mu_{A'A} \varepsilon_\mu$ with $\varepsilon_\mu = \sqrt{2}/2(1, 0, 0, 0, 0, 0, 0, -i)$, corresponding to $Y_i = 0$ in (3.8) below.

stronger. In section 3.4, we shall explain how it can be expressed as holomorphic geodesic motion on the twistor space Z. For what concerns the second condition $r' = N$, it is consistent with the condition $p_r = \pm 1$ following from the Hamiltonian constraint (2.27) at extremality, but implies that only the choice of the upper sign in this relation is consistent with supersymmetry.

2.4 Geodesics and one-parameter subgroups

Since the target space $\mathcal{M}_3^*$ is a symmetric space, all geodesics correspond to one-parameter subgroups in G_3. A geodesic passing through the point e_0 at $\tau = 0$ with initial velocity p_0 is given by

$$e(\tau) = k(\tau) \cdot e^{p_0 \tau/2} \cdot e_0 , \quad M(\tau) \equiv e^T(\tau) \cdot e(\tau) = e_0^T \cdot e^{p_0 \tau} \cdot e_0 \qquad (2.38)$$

where p_0 is a non-compact (i.e. symmetric) element in $\mathfrak{g}_3$, $k(\tau)$ is the unique element of K_3 which brings $e(\tau)$ back to the Iwasawa gauge, and τ is the affine parameter. The $\mathfrak{g}_3$-valued conserved charge inherited from the right action of G_3 is then given by

$$Q = -dM \, M^{-1} = -e_0^t \, p_0 \, e_0^{-t} . \qquad (2.39)$$

The velocity p_0 may be traded for the Noether charge Q, but it should be noted that the latter cannot be chosen independently from the initial position M_0, since $QM = MQ^T$ at all times. In terms of Q, the geodesic motion is given by

$$e(\tau) = k(\tau) \cdot e_0 \cdot e^{-Q^T \tau/2} , \quad M(\tau) = e^{-Q\tau/2} \cdot M_0 \cdot e^{-Q^T \tau/2} \qquad (2.40)$$

The affine parameter τ is equal to the radial parameter ρ in the gauge $N(\rho) = r^2(\rho)$. The motion of $r(\rho)$ may be obtained by integrating the Hamiltonian constraint (2.27), and depends only on p_0^2.

The action of an element g of G_3 takes the solution (2.40) to another solution with $Q \to g^T Q g^{-T}, M_0 \to g^T M_0 g$. As a result, trajectories may be classified according to the orbit of the matrix of Noether charges Q under the co-adjoint action of G_3. Of special interest are nilpotent orbits, i.e. those for which $Q^r = 0, Q^{r-1} \neq 0$ for some $r \geq 2$ (the degree r depends on representation in which Q is evaluated; here we consider the defining representation of G_3). Indeed, it was pointed out in [21] that BPS black holes in very special $\mathcal{N} = 2$ supergravity theories correspond to specific nilpotent orbits of degree 3. Subsequently, it was shown that for very special $\mathcal{N} = 2$ supergravity with one vector multiplet, nilpotent orbits of degree 3 yield (in general non-BPS) extremal black holes in 4 dimensions [6]. It is straightforward to check that the argument in [6] extends to the present case. It is therefore interesting to determine the allowed nilpotent orbits of degree 3 for $G_3 = \mathrm{SO}(8, n_v + 2)$.

Since Q is conjugate to $p_0 = \begin{pmatrix} 0 & p^{Aa} \\ p^{Aa} & 0 \end{pmatrix}$ in the basis (2.16), the condition $Q^3 = 0$ amounts to

$$p^{Aa} p^{Bb} p^{Cc} C_{AB} \, \delta_{bc} = 0 . \qquad (2.41)$$

This condition is clearly obeyed by BPS solutions, which satisfy the quadratic constraint (2.37). In fact, one may check explicitly that (for $n_v \geq 2$) the Noether charge for BPS solutions lies in the orbit $(3^4, 1^{n_v-2})$ of the complexified group $\mathrm{SO}(10 + n_v, \mathbb{C})$ (see appendix B for a review of general facts about nilpotent orbits, and a table of the low-dimensional nilpotent orbits of orthogonal groups). This follows from the fact, to be discussed in section 3, that BPS trajectories can be lifted to holomorphic geodesics on the twistor space Z, which is equal to the orbit $(3^4, 1^{n_v-2})$ via (B.1). This orbit is the "largest" nilpotent orbit of degree 3 (amongst orbits with dimension less than $\mathcal{O}(8n_v)$), in the sense that it intersects the closure of any orbit of degree 3 (as apparent on figure 1). This identification implies that the phase space of 1/4-BPS solutions in $\mathcal{N} = 4$ supergravity with n_v vector multiplets is $8n_v + 28$ dimensional,[5] much larger than the dimension $4n_v + 26$ of the phase space of 1/2-BPS solutions in a $\mathcal{N} = 2$ supergravity with the same number $n_v + 6$ of vector fields [5]. The extra degrees of freedom correspond to the n_v hypermultiplets coming from the decomposition of the n_v $\mathcal{N} = 4$ vector multiplets. The twistor techniques of the next section in principle allow to find the most general 1/4-BPS solution, although we fall short of this goal due to technical difficulties explained in section 3.4.

For what concerns non-BPS extremal black holes, they correspond to solutions of (2.41) which do not satisfy (2.37). Since there exist (at least) two different real nilpotent orbits of type $(3^4, 1^{n_v-2})$, related by an outer automorphism of $\mathrm{SO}(8, n_v + 2)$, it is natural to conjecture that such a transformation will map BPS solutions to non-BPS extremal solutions. Finding the general form of these non-BPS solutions is outside the scope of this paper.

3. Twistorial techniques for $\mathcal{N} = 4$ BPS black holes

We now return to the supersymmetry condition (2.36), and introduce geometric methods which allow to implement these constraints, both at a classical and quantum level in a convenient way. We work with the original Riemannian space (2.9), and perform analytic continuations at the end.

As emphasized in [5], it is expedient to eliminate the existence quantifier in (2.36) by enlarging the phase space with the complex Killing spinor ε_μ. Since the latter is always of zero norm and defined up to the action of $\mathbb{C}^\times$, it is best viewed as an element of the complex symmetric space

$$F = \frac{\mathrm{SO}(8, \mathbb{R})}{\mathrm{U}(4)} \sim \frac{\mathrm{SO}(8, \mathbb{R})}{\mathrm{SO}(2) \times \mathrm{SO}(6)} \tag{3.1}$$

As we explain in more detail below, this equality reflects the fact that Cartan pure spinors in 8 dimensions are just zero norm spinors. Remarkably, it is possible to fiber[6] F over $\mathcal{M}_3$ such that the total space Z admits an integrable complex structure [12]: this is achieved by "cancelling the SO(8) factors", namely by considering the homogeneous (but not sym-

---

[5]This is before enforcing the first class constraint $K = 0$.

[6]Note that unlike the quaternionic-Kähler case, the fiber is *not* the sphere of almost complex structures S^6, but a complexification thereof

metric) complex space $Z \equiv M_3 \backslash G_3$

$$Z = \frac{\mathrm{SO}(8, n_v + 2, \mathbb{R})}{\mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SO}(n_v + 2)} \sim \frac{\mathrm{SO}(8, n_v + 2, \mathbb{R})}{\mathrm{SO}(2) \times \mathrm{SO}(6) \times \mathrm{SO}(n_v + 2)} \ . \tag{3.2}$$

The integrable complex structure is afforded by the $\mathrm{U}(1) = \mathrm{SO}(2)$ factor in the denominator. Moreover, as we show below, the BPS conditions (2.36) guarantee that the geodesic motion on $\mathcal{M}_3$ can be lifted to a holomorphic curve on Z. This construction parallels the $\mathcal{N} = 2$ case discussed in [3], upon replacing the complex projective twistor line $\mathbb{C}P^1$ with the Grassmannian F.

3.1 Parametrizing the fiber

In a basis where the invariant metric takes the off-diagonal block form $\eta_8 = \begin{pmatrix} & \mathbb{I}_4 \\ \mathbb{I}_4 & \end{pmatrix}$, a coset representative of $\mathrm{SO}(8, \mathbb{R})$ may be chosen as

$$e_F = \begin{pmatrix} \mathbb{I}_4 & 0 \\ \bar{X} & \mathbb{I}_4 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{1 - X\bar{X}} & 0 \\ 0 & \sqrt{1 - \bar{X}X} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_4 & X \\ 0 & \mathbb{I}_4 \end{pmatrix} \ . \tag{3.3}$$

where X_{IJ} $(I, J = 1 \dots 4)$ is a 4×4 antisymmetric complex[7] matrix X. This decomposition realizes the Harish-Chandra embedding $K \backslash G(\mathbb{R}) \hookrightarrow P(\mathbb{C}) \backslash G(\mathbb{C})$ where $G(\mathbb{R}) = \mathrm{SO}(8, \mathbb{R})$ and $P(\mathbb{C})$ is the parabolic subgroup of lower block-triangular matrices of the form $\begin{pmatrix} * & \\ * & * \end{pmatrix}$, and guarantees that X are complex coordinates on F. Moreover, it makes explicit the holomorphic action of $G(\mathbb{C})$ on F, by right multiplication on (3.5) followed by left multiplication by an element of $P(\mathbb{C})$. On general grounds [26], a Kähler potential for the invariant metric on F is given by the logarithm of a character of $K(\mathbb{C}) = GL(4, \mathbb{C})$ evaluated on the block-diagonal component in the Harish-Chandra decomposition (3.3),

$$K(X, \bar{X}) = \log \det(1 - X\bar{X}) \ . \tag{3.4}$$

The first four rows of the right-most matrix in (3.3) define an isotropic[8] 4-plane $\mathbb{C}^4 = (\mathbb{I}_4 \,|\, X)$ inside $\mathbb{C}^8$. Such isotropic planes are also known as projectivized pure spinors in Cartan's sense.

On the other hand, in a basis where the invariant metric takes the form $\tilde{\eta}_8 = \begin{pmatrix} & & 1 \\ & \mathbb{I}_6 & \\ 1 & & \end{pmatrix}$, a coset representative of $\mathrm{SO}(8, \mathbb{R})/\mathrm{SO}(2) \times \mathrm{SO}(6)$ may be chosen as

$$\tilde{e}_F = \begin{pmatrix} 1 & & \\ \bar{Y}^i & \mathbb{I}_6 & \\ -\frac{1}{2} \sum \bar{Y}_k^2 & -\bar{Y}_i & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-K(Y, \bar{Y})/2} & & \\ & [A(Y, \bar{Y})]^{-1/2} & \\ & & e^{K(Y, \bar{Y})/2} \end{pmatrix} \cdot \begin{pmatrix} 1 & Y_i & -\frac{1}{2} \sum Y_k^2 \\ & \mathbb{I}_6 & -Y_i \\ & & 1 \end{pmatrix} \tag{3.5}$$

---

[7]Since we are dealing with the compact form of $\mathrm{SO}(8)$, the matrix representation in this basis has to be complex. The split form $\mathrm{SO}(4, 4)$ would instead be obtained by taking X and $\bar{X}$ as independent real variables.

[8]i.e. a 4-plane of zero norm vectors: $(\mathbb{I}_4 \,|\, X)^T \eta_8 (\mathbb{I}_4 \,|\, X) = 0$ since $X = -X^T$.

where the scalar $e^{K(Y,\bar{Y})}$ and the 6×6 matrix $A(Y,\bar{Y})$ are determined in terms of the complex coordinates Y_i and their complex conjugate $\bar{Y}_i$:

$$K(Y,\bar{Y}) = \frac{1}{2}\log\left(1 + \sum_i Y_i\bar{Y}_i + \frac{1}{4}\left(\sum_i Y_i^2\right)\left(\sum_i \bar{Y}_i^2\right)\right), \tag{3.6}$$

$$A_{ij}(Y,\bar{Y}) = \delta_{ij} + e^K\left[Y_i\bar{Y}_j - Y_j\bar{Y}_i - \frac{1}{2}Y_iY_j\left(\sum \bar{Y}_k^2\right)\right.$$
$$\left. -\frac{1}{2}\bar{Y}_i\bar{Y}_j\left(\sum Y_k^2\right) + Y_i\bar{Y}_j\left(\sum Y_k\bar{Y}_k\right)\right]. \tag{3.7}$$

Again, $K(Y,\bar{Y})$ provides the Kähler potential for the SO(8)-invariant Kähler metric on F. This time, the first row $(1, Y_i, -\frac{1}{2}\sum Y_k^2)$ in the right-most matrix in (3.5) provides the most general null vector for the metric η_8, up to a $\mathbb{C}^\times$ action. Thus, in eight dimensions Cartan pure spinors are indeed the same as projectivized null vectors.

Based on this observation, it is natural to identify this null vector with the Killing spinor ε_μ,

$$\varepsilon_\mu = \left(\frac{2 - \sum Y_k^2}{2\sqrt{2}}, Y_i, \frac{2 + \sum Y_k^2}{2i\sqrt{2}}\right), \quad \sum_\mu \varepsilon_\mu^2 = 0 . \tag{3.8}$$

To see the relation to the coordinates X_{IJ} note that for a fixed null vector ε_μ, the equations

$$\forall A', \quad \varepsilon_\mu \Gamma^\mu_{A'A} p^A = 0 \tag{3.9}$$

select an isotropic 4-plane in the 8-dimensional space of the spinors p^A. In fact, using the explicit representation of the SO(8) Dirac matrices given in appendix A, we have the rank 4 matrix

$$\varepsilon_\mu \Gamma^\mu_{A'A} = \begin{pmatrix} i\sqrt{2}\,\mathbb{I}_4 & \sum_k Y_k\Sigma^k \\ \sum_k \tilde{Y}_k\Sigma^i & -\frac{i}{\sqrt{2}}\sum_k Y_k^2 \end{pmatrix} \tag{3.10}$$

where Σ_i are SO(6) Sigma matrices. Identifying the first four rows of this matrix with the isotropic 4-plane $(\mathbb{I}_4 \,|\, X)_{IA}$ leads to the relation between the X and Y coordinates,

$$X_{IJ} = -\frac{i}{\sqrt{2}}Y_i \Sigma^i_{IJ} . \tag{3.11}$$

It may be checked explicitly that the Kähler potentials (3.6) and (3.4) agree, up to a Kähler transformation.

3.2 Bottom-up construction of the twistor space

The homogeneous complex space Z defined in (3.2) may be parameterized by relaxing the Iwasawa gauge in (2.11), and introducing a coset representative $\hat{e}_F$ of the fiber U(4)\SO(8),

$$e_Z = \hat{e}_F \cdot e_{8,n_v+2} , \tag{3.12}$$

where $\hat{e}_F$ is obtained by embedding e_F inside the maximal compact subgroup SO(8) $\times$ SO($n_v + 2$) of G_3,

$$\hat{e}_F = \Omega \cdot \begin{pmatrix} \mathbb{I}_4 & 0 & 0 \\ 0 & \mathbb{I}_{n_v+2} & 0 \\ \bar{X} & 0 & \mathbb{I}_4 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{1-X\bar{X}} & 0 & 0 \\ 0 & \mathbb{I}_{n_v+2} & 0 \\ 0 & 0 & \sqrt{1-\bar{X}X} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_4 & 0 & X \\ 0 & \mathbb{I}_{n_v+2} & 0 \\ 0 & 0 & \mathbb{I}_4 \end{pmatrix} \cdot \Omega^{-1} . \tag{3.13}$$

Here, the matrix

$$
\Omega = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{2} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\
0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\
0 & 0 & 0 & 0 & \sqrt{2}\,\mathbb{I}_{n_v} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{i}{2} & 0 & 0 & 0
\end{pmatrix} \tag{3.14}
$$

provides the change of basis from the metric (2.10) to the metric

$$
\hat{\eta}_{8,n_v+2} = \begin{pmatrix}
 & & \mathbb{I}_4 \\
 & -\mathbb{I}_{n_v+2} & \\
\mathbb{I}_4 & &
\end{pmatrix} = \Omega^T\,\eta_{8,n_v+2}\,\Omega \tag{3.15}
$$

Block-diagonal matrices of the form

$$
\begin{pmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & A^{-T}
\end{pmatrix}, \quad A \in \mathrm{U}(4),\ , B \in \mathrm{SO}(n_v+2) \tag{3.16}
$$

generate a $\mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SO}(n_v+2)$ subgroup of the maximal compact subgroup $\mathrm{SO}(8) \times \mathrm{SO}(n_v+2)$. It is important to note that the $\mathrm{U}(4)$ factor inside $\mathrm{SO}(8)$ is distinct from the $\mathrm{U}(1) \times \mathrm{SO}(6)$ 4-dimensional R-symmetry group. These two groups only share a $\mathrm{U}(3)$ common subgroup, which is manifest with our choice of Ω in (3.14).

Now, consider the decomposition of the Lie algebra of $\mathrm{SO}(8, n_v+2)$ under $\mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SO}(n_v+2)$,

$$
\bar{6}|_{-2} \oplus (\bar{4}, n_v+2)|_{-1} \oplus [\mathrm{SU}(4) \oplus \mathrm{U}(1) \oplus \mathrm{SO}(n_v+2)]_0 \oplus (4, n_v+2)|_1 \oplus 6|_2 \tag{3.17}
$$

where the subscript indicates the $\mathrm{U}(1)$ charge. Here, in contrast to the 5-grading (2.12), the Cartan involution exchanges the spaces of positive and negative charge. The right-invariant one-form $\theta^Z = \Omega^{-1} \cdot de_Z e_Z^{-1} \cdot \Omega$ decomposes along each summand in (3.17) as

$$
\theta^Z = \bar{\theta}^Z_{IJ} \bar{T}^{IJ} + \bar{\theta}^Z_{Ia} \bar{T}^{Ia} + (\theta^Z_{\mathrm{SU}(4)} + \theta^Z_{\mathrm{U}(1)} + \theta^Z_{\mathrm{SO}(n_v+2)}) + \theta^Z_{Ia} T^{Ia} + \theta^Z_{IJ} T^{IJ} \tag{3.18}
$$

where the generators $\bar{T}^{IJ}, \bar{T}^{Ia}, T^{Ia}, T^{IJ}$ have charge $-2, -1, 1, 2$ respectively. On general grounds, the positive charge components $\theta_{IJ}, \theta^Z_{Ia}$ correspond to (1,0) forms on Z, while their complex conjugate $\bar{\theta}^Z_{IJ}, \bar{\theta}^Z_{Ia}$ are (0,1) forms. In the basis corresponding to the metric (3.15),

the U(1) factor is generated by the diagonal matrix $\text{diag}(\mathbb{I}_4, 0_{n_v+2}, -\mathbb{I}_4)$, and therefore $\theta^Z_{IJ}, \theta^Z_{Ia}$ are just the 4×4 and $4 \times (n_v + 2)$ blocks in the upper triangular part of θ_Z,

$$\theta^Z = \begin{pmatrix} \theta^Z_{\mathrm{SU}(4)} + \theta^Z_{\mathrm{U}(1)} & \theta^Z_{Ia} & \theta^Z_{IJ} \\ \bar{\theta}^Z_{Ia} & \theta^Z_{\mathrm{SO}(n_v+2)} & \theta^Z_{Ia} \\ \bar{\theta}^Z_{IJ}, & \bar{\theta}^Z_{Ia} & -(\theta^Z_{\mathrm{SU}(4)})^\dagger - \theta^Z_{\mathrm{U}(1)} \end{pmatrix} \tag{3.19}$$

In terms of the components p_{Aa}, θ_{AB} of the right-invariant one-form (2.16) on the base $\mathcal{M}_3$, the (1,0) forms read

$$\theta^Z_{Ia} = V(\mathbb{I}_4 \,|X)_{IA} \, p^{Aa} \,, \tag{3.20}$$

$$\theta^Z_{IJ} = V \left(dX + \theta^{(2)} + \theta^{(4)} X - X \theta^{(1)} - X \theta^{(3)} X \right) V^T \tag{3.21}$$

where

$$V = (1 - X\bar{X})^{-1/2} \tag{3.22}$$

and $\theta^{(k)}$ are the 4×4 blocks in the SO(8) connection,

$$\theta_{AB} = \begin{pmatrix} \theta^{(1)}_{IJ} & \theta^{(2)}_{IJ} \\ \theta^{(3)}_{IJ} & \theta^{(4)}_{IJ} \end{pmatrix} \,. \tag{3.23}$$

Similarly, the $(0, 1)$ invariant forms may be obtained from the lower triangular part of θ_Z, or by complex conjugation from the $(1, 0)$ forms, using the fact that $\bar{\theta}^{(2)} = \theta^{(3)}, \bar{\theta}^{(1)} = \theta^{(4)}$:

$$\bar{\theta}^{Ia}_Z = \bar{V}(\bar{X} \,|\mathbb{I}_4)_{IA} \, p^{Aa} \,, \tag{3.24}$$

$$\bar{\theta}^{IJ}_Z = \bar{V} \left(d\bar{X} + \theta^{(3)} + \theta^{(1)} \bar{X} - \bar{X} \theta^{(4)} - \bar{X} \theta^{(2)} \bar{X} \right) \bar{V}^T \equiv d\bar{X} + \overline{\mathcal{P}} \tag{3.25}$$

Giving the (1,0) and (0,1) forms uniquely specify an almost complex structure $\mathcal{J}$ on Z. Since linear combinations of (1,0) forms stay of (1,0) type, we may set $V = 1$ in (3.20) and (3.24) , and take as a basis of (1,0) forms

$$DX_{IJ} \equiv dX + \theta^{(2)} + \theta^{(4)} X - X \theta^{(1)} - X \theta^{(3)} X \equiv dX + \mathcal{P} \,, \tag{3.26}$$

$$DZ_{Ia} \equiv (\mathbb{I}_4 \,|X)_{IA} \, p^{Aa} \,. \tag{3.27}$$

In the next subsection, we shall show that $\mathcal{J}$ is in fact integrable. Observe that the (1,0)-forms DZ_{Ia} are linear combinations of the cotangent forms p^{Aa} whose coefficients are holomorphic functions on the fiber, while the (1,0)-forms DX are obtained by adding the "projectivized SO(8) connection" $\mathcal{P}$ to the holomorphic differentials dX on the fiber. This directly parallels the twistor construction for quaternionic-Kähler spaces.

Finally, a family of invariant Hermitian metrics on Z may be constructed by forming $\mathrm{SU}(4) \times \mathrm{SO}(n_v + 2)$ invariant quadratic combinations of the (1,0) and (0,1) forms,

$$ds^2 = \theta^Z_{IJ} \bar{\theta}^{IJ}_Z + \nu \, \theta^Z_{Ia} \bar{\theta}^{Ia}_Z \,. \tag{3.28}$$

The parameter ν can be fixed by requiring that the metric is Kähler (see section 3.3).

3.3 Top-down construction of the twistor space

We now describe an alternative construction of Z, which makes it manifest that the almost complex structure $\mathcal{J}$ is integrable, and that Z admits an invariant Kähler metric. As in our discussion of the Kähler metric on the fiber in section 3.1, and in analogy with [13], we rely on the Harish-Chandra embedding $M_3 \backslash G_3(\mathbb{R}) \hookrightarrow P_3(\mathbb{C}) \backslash G_3(\mathbb{C})$ where $P_3(\mathbb{C})$ is the parabolic subgroup of lower block-triangular matrices in the basis where the metric takes the off-diagonal form

$$\eta_{8,n_v+2} = \begin{pmatrix} & & \mathbb{I}_4 \\ & -\mathbb{I}_{n_v+2} & \\ \mathbb{I}_4 & & \end{pmatrix}, \quad P_3(\mathbb{C}) = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix} \tag{3.29}$$

This embedding is achieved by decomposing any element $g \in G_3$ as a product

$$g = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_4 & \mathcal{Z} & \widetilde{\mathcal{X}} \\ 0 & \mathbb{I}_{n_v+2} & \mathcal{Z}^T \\ 0 & 0 & \mathbb{I}_4 \end{pmatrix} \tag{3.30}$$

where $\widetilde{\mathcal{X}} \equiv \mathcal{X} + \frac{1}{2} \mathcal{Z} \mathcal{Z}^T$ is an 4×4 antisymmetric complex matrix and $\mathcal{Z}$ is a $4 \times (n_v + 2)$ complex matrix. The map $g \in M_3(\mathbb{R}) \backslash G_3(\mathbb{R}) \hookrightarrow (\mathcal{X}, \mathcal{Z}) \in P_3(\mathbb{C}) \backslash G_3(\mathbb{C})$ is well-defined since a left-multiplication by an element of $K(\mathbb{R})$ only affects the lower triangular part of the decomposition, and it is injective since $P_3(\mathbb{C}) \backslash G_3(\mathbb{C}) \cap M_3(\mathbb{R})$ consists only of the identity. In particular, choosing $g = \Omega^{-1} e_Z \Omega$ where e_Z is the coset representative in (3.12), we can express $(\mathcal{X}, \mathcal{Z})$ as a function of the coordinates on the base $U, \tau^i, \bar{\tau}^i, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma$ and the complex coordinates X_{IJ} on the fiber (the resulting expressions turn out to be very cumbersome and are best omitted here). Note that $(\mathcal{X}, \mathcal{Z})$ is independent of $\bar{X}_{IJ}$, as the two $\bar{X}$-dependent factors in (3.13) only affect the lower triangular part. Thus, the Harish-Chandra embedding provides a holomorphic parametrization of the "twistor lines", i.e. the fibers of the projection $Z \to \mathcal{M}_3$. This map was also referred to as "the twistor map" in [3].

Conversely, an element of $(\mathcal{X}, \mathcal{Z}) \in P_3(\mathbb{C}) \backslash G_3(\mathbb{C})$ may be mapped into an element of $G_3(\mathbb{R})$

$$e_Z = \begin{pmatrix} \mathbb{I}_4 & 0 & 0 \\ \bar{\mathcal{Z}} & \mathbb{I}_{n_v+2} & 0 \\ \bar{\widetilde{\mathcal{X}}} & \bar{\mathcal{Z}}^T & \mathbb{I}_4 \end{pmatrix} \cdot \begin{pmatrix} A^{-1/2} & 0 & 0 \\ 0 & B^{-1/2} & 0 \\ 0 & 0 & (A^T)^{1/2} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_4 & \mathcal{Z} & \widetilde{\mathcal{X}} \\ 0 & \mathbb{I}_{n_v+2} & \mathcal{Z}^T \\ 0 & 0 & \mathbb{I}_4 \end{pmatrix} \tag{3.31}$$

where A and B are 4×4 and $(n_v + 2)^2$ matrices afforded by the decomposition

$$\begin{pmatrix} \mathbb{I}_4 & \mathcal{Z} & \widetilde{\mathcal{X}} \\ 0 & \mathbb{I}_{n_v+2} & \mathcal{Z}^T \\ 0 & 0 & \mathbb{I}_4 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{I}_4 & 0 & 0 \\ \bar{\mathcal{Z}} & \mathbb{I}_{n_v+2} & 0 \\ \bar{\widetilde{\mathcal{X}}} \bar{\mathcal{Z}} & \bar{\mathcal{Z}}^T & \mathbb{I}_4 \end{pmatrix} = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix} \cdot \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A^{-T} \end{pmatrix} \cdot \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} . \tag{3.32}$$

The Iwasawa decomposition of e_Z then allows to express the coordinates $U, \tau^i, \bar{\tau}^i, \zeta^\Lambda, \tilde{\zeta}_\Lambda,$ $\sigma, X, \bar{X}$ on $\mathcal{M}_3 \times F$ in terms of $(\mathcal{X}, \mathcal{Z})$ and their complex conjugates. This reciprocal map was termed "covariant c-map", or superconformal quotient, in [3]. Again, this map

is in principle computable, but the resulting expressions are too cumbersome to be of any practical use.

While only the real group $G_3(\mathbb{R})$ acts on the base $\mathcal{M}_3$, the action on the twistor space Z can be extended to the complexified group $G_3(\mathbb{C})$: it acts by right-multiplication on the coset representative (3.30), followed by a left-multiplication by an appropriate lower triangular matrix so as to return to the strictly upper triangular gauge. The complex coordinates $(\mathcal{X}, \mathcal{Z})$ are adapted to the holomorphic action of the nilpotent group of strictly upper-block diagonal matrices, in the sense that no compensating left-action is needed. This action is generated by the vector fields

$$E_I^a = \partial_{\mathcal{Z}^{Ia}} + \epsilon_{IJKL}\mathcal{Z}^{Ja}\partial_{\mathcal{X}_{KL}}, \quad E^{IJ} = \partial_{\mathcal{X}_{IJ}} \tag{3.33}$$

which satisfy the Heisenberg-type commutation relations

$$\left[E_I^a, E_J^b\right] = \epsilon_{IJKL}\,\delta_{ab}\,E^{KL}\ . \tag{3.34}$$

For applications to black hole physics, it would be desirable to have complex coordinates adapted to the Heisenberg algebra (2.29), which corresponds to the electric, magnetic and NUT charges. As for the $SU(2,1)$ case studied in [13], it should be possible to obtain this change of variable by taking the limit $U \to -\infty, \tau_2 \to 0$ in the twistor map. We note however that for $n_v = 0$, there is an obvious holomorphic action of $G_3(\mathbb{R})$ on 14 complex variables, adapted to Heisenberg algebra (2.29), corresponding to the "fake" Harish-Chandra decomposition in the original basis (2.10),

$$\begin{pmatrix} * & & & & \\ * & * & & & \\ * & * & * & & \\ * & * & * & * & * \\ * & * & * & & * \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta & \xi & -\frac{1}{2}\xi\xi^t & \alpha - \frac{1}{2}\xi\tilde{\xi}^t \\ 0 & 1 & \tilde{\xi} & -\alpha - \frac{1}{2}\xi\tilde{\xi}^t & -\frac{1}{2}\tilde{\xi}\tilde{\xi}^t \\ & & \mathbb{I}_6 & -\xi^t & -\tilde{\xi}^t \\ & & & 1 & 0 \\ & & & -\beta & 1 \end{pmatrix} \tag{3.35}$$

where α and β are two complex variables, and ξ^Λ, $\tilde{\xi}_\Lambda$ are two complex vectors in $\mathbb{C}^6$. It would be interesting to find the change of variable from the complex coordinates $(\mathcal{X}, \mathcal{Z})$ to $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha, \beta)$.

An Hermitian metric on Z can be obtained by computing the right-invariant form, projecting out the $M_3(\mathbb{C})$ part, and taking M_3-invariant quadratic combinations as in (3.28). The strictly upper-triangular components of $dg \cdot g^{-1}$ provide right-invariant (1,0) forms

$$\theta_{IJ}^Z = \tilde{V}\left(d\mathcal{X} + \frac{1}{2}\mathcal{Z}d\mathcal{Z}^T - \frac{1}{2}d\mathcal{Z}\mathcal{Z}^T\right)\tilde{V}^T, \quad \theta_{Ia}^Z = \tilde{V}d\mathcal{Z}\ . \tag{3.36}$$

where

$$\tilde{V} = A^{-1/2} \tag{3.37}$$

Setting $\tilde{V} = 1$ in (3.36), we obtain a basis of holomorphic (i.e. $\bar{\partial}$-closed) (1,0) forms,

$$D\mathcal{X} \equiv d\mathcal{X} + \frac{1}{2}\left(\mathcal{Z}d\mathcal{Z}^T - d\mathcal{Z}\mathcal{Z}^T\right), \quad DZ = d\mathcal{Z}\ . \tag{3.38}$$

The antisymmetric matrix of holomorphic 1-forms $D\mathcal{X}$ plays the rôle of the holomorphic contact distribution in the quaternionic-Kähler case. Note that under a right-action of $G(\mathbb{C})$, $D\mathcal{X}_{IJ}$ transforms by an element of $GL(4,\mathbb{C})$, corresponding to the block diagonal component of the compensating lower triangular matrix required to restore the upper triangular gauge. In the quaternionic-Kähler case, this issue can be circumvented by introducing a new $\mathbb{C}^{\times}$ valued variable t, and considering the one-form $tD\mathcal{X}$: the rescaling of DX can be reabsorbed by a rescaling of t, leading to a globally defined holomorphic one-form on $\mathbb{C}^{\times} \times Z$, whose exterior derivative is the holomorphic two-form Ω on the hyperkähler cone of $\mathcal{M}_3$ [27]. In the present case, one may similarly introduce 6 new variables t^{IJ} and consider the globally defined holomorphic two-form $\Omega = d(t^{IJ}X_{IJ})$. We shall return to this possibility momentarily.

In the above construction of the metric (3.28), it is difficult to fix the coefficient ν such that the metric is Kähler. However, according to the general prescription of [26], we know that a Kähler potential for an invariant metric on Z is given by the logarithm of a character of $M_3(\mathbb{C})$ evaluated on the block diagonal part in the decomposition (3.32):[9]

$$K_Z(\mathcal{X}, \mathcal{Z}, \bar{\mathcal{X}}, \bar{\mathcal{Z}}) = \log \det \left[\mathbb{I}_4 + \mathcal{Z}\bar{\mathcal{Z}} + \widetilde{\mathcal{X}}\bar{\widetilde{\mathcal{X}}}^T \right] \tag{3.39}$$

Comparison to the metric (3.28) fixes $\nu = -1$. It would be interesting to check whether the metric is Kähler-Einstein, as in the case of twistor spaces of quaternionic-Kähler spaces.

Given the transformation properties of the kernel matrix $A(\mathcal{X}, \mathcal{Z}, \bar{\mathcal{X}}, \bar{\mathcal{Z}})$, it is also natural to consider higher dimensional spaces with Kähler potential

$$K(t, \mathcal{X}, \mathcal{Z}, \bar{t}, \bar{\mathcal{X}}, \bar{\mathcal{Z}}) = \bar{t} \cdot R \left[\mathbb{I}_4 + \mathcal{Z}\bar{\mathcal{Z}} + \widetilde{\mathcal{X}}\bar{\widetilde{\mathcal{X}}}^T \right] \cdot t \tag{3.40}$$

where t transforms in some finite dimensional representation R of $GL(4,\mathbb{C})$. For $t = t^{IJ}$ in the 6-dimensional antisymmetric representation, combining this result with the construction in the paragraph below (3.38), we obtain a $8(n_v+5)$ real-dimensional Kähler space with a (2,0) holomorphic form and a homothetic Killing vector. It is natural to conjecture that this provides the natural hyperkähler metric [28] on a complex nilpotent co-adjoint orbit of $SO(n_v + 10, \mathbb{C})$ associated to the partition $(2^4, 1^{n_v+2})$, of complex dimension $4(n_v + 5)$.

3.4 Supersymmetry and holomorphy

We now return to the physical motivation for this geometric construction, the supersymmetry conditions (2.36). As we discussed below (3.9), there is an equivalence

$$\varepsilon_\mu \, \Gamma^\mu_{A'A} \, p^{Aa} = 0 \quad \Leftrightarrow \quad (\mathbb{I}_4|X)_{IA} \, p^{Aa} = 0 \,, \tag{3.41}$$

provided the null vector ε_μ is related to X_{IJ} via (3.8), (3.11). Moreover, in (3.26), we have established that the one-forms $DZ_{Ia} = (\mathbb{I}_4|X)_{IA}p^{Aa}$ are (1,0) forms with respect to the complex structure on Z. Therefore, if we lift the geodesic motion on $\mathcal{M}_3$ to the

---

[9]Note that K_Z reduces to the Kähler potential (3.4) on the fiber F at $\mathcal{Z} = 0$.

twistor space Z by requiring that at every point, $DX_{IJ} = 0$, we conclude that supersymmetric geodesics on $\mathcal{M}_3$ have a tangent vector of type (0,1) at every point, and therefore correspond to an anti-holomorphic curves $\rho : \mathbb{C} \to Z$. In practice, this means that the holomorphic coordinates z^i are constant along the flow, while the anti-holomorphic coordinates $\bar{z}^{\bar{i}}$ evolve[10] in such a way that the gradient of the Kähler potential grows linearly with the affine parameter [3][11]

$$\partial_{z^i} K = c_i \tau + d_i . \tag{3.42}$$

Moreover, the BPS constraints (2.37), re-expressed as $DZ_{Ia} = DX_{IJ} = 0$, now manifestly form a system of first class constraints, as the Lie bracket of two (anti)holomorphic vectors is necessarily (anti)holomorphic. As in the $\mathcal{N} = 2$ case [5], we can therefore identify the 1/4-BPS phase space as the twistor space Z, equipped with its Kähler form.

In order to make the best use of this geometric statement, it would be desirable to construct a coordinate system on Z adapted to the Heisenberg symmetries (2.29). This would enable us to determine the most general 1/4-BPS spherically symmetric solutions in $\mathcal{N} = 4$ supergravity, and also to compute the exact BPS black hole wave function as a Penrose transform of a holomorphic wave-function on Z, along the lines of [3]. While we have been unable to carry out this computation, in the next subsection we construct some holomorphic functions on Z which provide BPS wave functions for solutions with certain charges.

3.5 Some holomorphic functions on Z

In this section, we construct some holomorphic functions on Z in the coordinate system $U, \tau^i, \bar{\tau}^i, \zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma, X, \bar{X}$ adapted to the fibration $F \to Z \to \mathcal{M}_3$. For ease of notation, we denote the entries in X_{IJ} as

$$X = \begin{pmatrix} 0 & y_1 & y_2 & y_3 \\ -y_1 & 0 & x_3 & -x_2 \\ -y_2 & -x_3 & 0 & x_1 \\ -y_3 & x_2 & -x_1 & 0 \end{pmatrix} \tag{3.43}$$

and similarly for $\bar{X}$.

Our first observation is that x_1, x_2, x_3 and $y_2/y_1, y_3/y_1$ are holomorphic functions on Z. This follows from the fact that their differentials are of (1,0) type,

$$dx_1 = Dx_1 + \frac{1}{\sqrt{2}} \left(-y_3 DZ_{31} + iy_3 DZ_{32} + y_2 DZ_{41} - iy_2 DZ_{42} \right) \tag{3.44}$$

$$dx_2 = Dx_2 + \frac{1}{\sqrt{2}} \left(y_3 DZ_{21} - iy_3 DZ_{22} - y_1 DZ_{41} + iy_1 DZ_{4,2} \right)$$

$$dx_3 = Dx_3 + \frac{1}{\sqrt{2}} \left(-y_2 DZ_{21} + iy_2 DZ_{22} + y_1 DZ_{31} - iy_1 DZ_{32} \right)$$

---

[10]Due to the analytic continuation from $\mathcal{M}_3$ to $\mathcal{M}_3^*$, the complex coordinates z^i should be treated as independent variables.

[11]This follows directly from the geodesic equation $\ddot{z}^i + \Gamma^i_{jk} \dot{z}^j \dot{z}^k = 0$, given that the Christoffel symbol Γ^i_{jk} has no mixed holomorphic/anti-holomorphic components.

$$d\left(\frac{y_2}{y_1}\right) = \frac{2y_1 Dy_2 - 2y_2 Dy_1 + \sqrt{2}(y_2(DZ_{21} + iDZ_{22}) - y_1(DZ_{31} + iDZ_{32}))}{2y_1^2}$$

$$d\left(\frac{y_3}{y_1}\right) = \frac{2y_1 Dy_3 - 2y_3 Dy_1 + \sqrt{2}(y_3(DZ_{21} + iDZ_{22}) - y_1(DZ_{41} + iDZ_{42}))}{2y_1^2}$$

Secondly, we note that the contraction of any Killing vector $\kappa^m \partial_{\phi^m}$ with the holomorphic contact distribution $D\mathcal{X}_{IJ}$ yields a 4×4 antisymmetric matrix of holomorphic functions, since $G_3(\mathbb{R})$ acts holomorphically on Z. Moreover the one forms, $D\mathcal{X}_{IJ}$ and $DX_{IJ} = DX_{IJ,m}d\phi^m$ are related to each other by a $GL(4,\mathbb{C})$ transformation,

$$DX = V^{-1}\tilde{V} \cdot D\mathcal{X} \cdot \tilde{V}^T V^{-T} . \tag{3.45}$$

Thus, for two Killing vectors κ^m and κ'^m, the combination

$$\langle \kappa, \kappa' \rangle \equiv \epsilon^{IJKL} \kappa^m DX_{IJ,m} \, \kappa'^n DX_{KL,n} \tag{3.46}$$

is holomorphic, up to an overall factor independent of κ and κ'. It may be checked explicitly that the product

$$y_1^{-1} e^{-2U} \langle \kappa, \kappa' \rangle \tag{3.47}$$

is holomorphic for one choice of κ and κ', and therefore for any pair of Killing vectors. Different pairs (κ, κ') may not necessary give independent holomorphic functions however: for $n_v = 0$, an explicit computation shows that a linear basis of holomorphic functions obtained in this way, using the Killing vectors $P^\Lambda, Q_\Lambda, K, H, Y_0, Y_\pm$ introduced in section 2.3, may be chosen as

$$\langle P^\Lambda, H \rangle , \quad \langle Q_\Lambda, H \rangle , \quad \langle P^\Lambda, Q_\Sigma \rangle = -\langle Q_\Lambda, P^\Sigma \rangle , \quad \langle H, H \rangle . \tag{3.48}$$

The remaining non-vanishing inner products can be expressed in terms of this basis as

$$\langle P^\Lambda, Y_0 \rangle = \langle P^\Lambda, H \rangle , \quad \langle Q_\Lambda, Y_0 \rangle = -\langle Q_\Lambda, H \rangle , \quad \langle Y_0, Y_0 \rangle = \langle Y_+, Y_- \rangle = -\langle H, H \rangle \tag{3.49}$$

This provides non-trivial examples of holomorphic functions on Z. Unfortunately, we have not managed to find eigenfunctions of the charge generators P^Λ, Q_Λ, K. Instead, one may check that the action of the Killing vectors on the holomorphic functions (3.48) is given by

$$P^\Lambda \cdot \langle P^\Sigma, H \rangle = 0 , \qquad Q_\Lambda \cdot \langle Q_\Sigma, H \rangle = 0 , \qquad K \cdot \langle P^\Lambda, H \rangle = 0 , \tag{3.50}$$

$$K \cdot \langle Q_\Sigma, H \rangle = 0 , \qquad Y_+ \cdot \langle P^\Lambda, H \rangle = 0 , \qquad Y_+ \cdot \langle Q_\Sigma, H \rangle = 0 , \tag{3.51}$$

$$P^\Lambda \cdot \langle Q_\Sigma, H \rangle = \langle P^\Lambda, Q_\Sigma \rangle , \quad Q_\Lambda \cdot \langle P^\Sigma, H \rangle = -\langle P^\Lambda, Q_\Sigma \rangle . \tag{3.52}$$

4. Discussion

In this work, we have analyzed 1/4-BPS spherically symmetric, stationary configurations in $D = 4, \mathcal{N} = 4$ supergravity, by dimensional reduction to one (radial) dimension. In parallel with the treatment of BPS black holes in $D = 4, \mathcal{N} = 2$ supergravity [3, 5], we have shown that such configurations correspond to supersymmetric geodesics on the three-dimensional

symmetric moduli space $\mathcal{M}_3$. This provides a powerful technique for obtaining new black hole solutions in 4 dimensions. Indeed, we have found that the phase space of BPS solutions is given by a degree 3 nilpotent orbit in $SO(8, 2 + n_v)$, whose real dimension $8n_v + 28$ is twice as large as expected by extrapolating the results for $\mathcal{N} = 2$ black holes. We have also found indications that the phase space of non-BPS extremal black holes is given by a nilpotent orbit with the same complexification as in the BPS case, but related by an outer automorphism of the real group $SO(8, 2+n_v)$. It would be interesting to study this further.

In addition, we have shown that supersymmetric geodesics on $\mathcal{M}_3$ can be lifted to holomorphic curves on a homogeneous complex space Z, the twistor space (3.2). In contrast to the $\mathcal{N} = 2$ case, the fiber does not parametrize the sphere of complex structures S^6, but rather the space $SO(8)/U(4)$ of isotropic 4-planes in $\mathbb{C}^8$. Moreover, Z does not carry a holomorphic contact form, but rather a 4×4 antisymmetric matrix of holomorphic contact forms. This complication has so far prevented us from constructing complex coordinates adapted to the Heisenberg symmetries of the problem, which were instrumental in [3, 5] for obtaining the BPS radial wave function for a black hole with fixed electric and magnetic charges. Nevertheless, there is no doubt that such a system can be constructed, and that a Penrose-type correspondence can be set up between holomorphic functions on Z and solutions of the second order partial differential equation $(C_{AB}\nabla^{Aa}\nabla^{Bb} + \lambda\delta^{ab})\Psi = 0$, which follows by quantizing (2.37). Irrespective of applications to black hole physics, this correspondence may be used to compute instanton corrections in 3 dimensions, provided one can identify a coupling in the low energy effective action governed by the same partial differential equation.

Acknowledgments

A. SO(8) gamma matrices

In this section, we describe our conventions for the SO(6) and SO(8) Dirac matrices used in the text. We start with the 4×4 SO(6) Sigma matrices Σ_i $(i = 1 \dots 6)$

$$
\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad
\Sigma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad
\Sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \text{(A.1)}
$$

$$
\Sigma_4 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad
\Sigma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad
\Sigma_6 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \text{(A.2)}
$$

corresponding to the SO(6) Dirac matrices in the Weyl representation,

$$\Gamma_i = \begin{pmatrix} 0 & \tilde{\Sigma}_i \\ \Sigma_i & 0 \end{pmatrix}, \quad \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}\mathbb{I}_8 \tag{A.3}$$

where $\tilde{\Sigma}_i = (-1)^i \Sigma_i = \Sigma_i^\dagger$. This is extended to a representation of the Clifford algebra of SO(7) by adding $\Gamma_7 = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6$. The charge conjugation matrix is given by

$$C = -i\Gamma_2\Gamma_4\Gamma_6 = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix}, \quad C\Gamma_i^T C^{-1} = -\Gamma_i. \tag{A.4}$$

The matrices Γ_i, Γ_7 supplemented with $\Gamma_0 = i\mathbb{I}_8$, can then serve as Sigma matrices[12] $\Gamma_{A'A}^\mu$ ($\mu = 0\dots 7$) for a chiral representation of the Clifford algebra of SO(8). In particular, the Lorentz generators $\Gamma_{AB}^{\mu\nu}$ in the spin 8_S representation of SO(8) are given by

$$\Gamma^{\mu\nu} = [\Gamma^\mu, \Gamma^\nu], \quad \Gamma^{\mu 0} = 2i\Gamma^\mu, \quad \Gamma^{0\mu} = -2i\Gamma^\mu \quad (\mu, \nu \neq 0) \tag{A.5}$$

satisfying the SO(8) algebra,

$$[\Gamma^{\mu\nu}, \Gamma^{\rho\sigma}] = -4\left(\delta^{\mu\rho}\Gamma^{\nu\sigma} + \delta^{\nu\sigma}\Gamma^{\mu\rho} - \delta^{\nu\rho}\Gamma^{\mu\sigma} - \delta^{\mu\sigma}\Gamma^{\nu\rho}\right) \tag{A.6}$$

Similarly, the Lorentz generators $\tilde{\Gamma}_{AB}^{\mu\nu}$ in the spin 8_C representation of SO(8) can be constructed as

$$\tilde{\Gamma}_i = \begin{pmatrix} 0 & \Sigma_i \\ \tilde{\Sigma}_i & 0 \end{pmatrix}, \quad \tilde{\Gamma}_0 = i\mathbb{I}_8, \quad \tilde{\Gamma}_7 = i\tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}_3\tilde{\Gamma}_4\tilde{\Gamma}_5\tilde{\Gamma}_6 \tag{A.7}$$

$$\tilde{C} = -i\tilde{\Gamma}_2\tilde{\Gamma}_4\tilde{\Gamma}_6 = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix} \tag{A.8}$$

$$\tilde{\Gamma}^{\mu\nu} = [\tilde{\Gamma}^\mu, \tilde{\Gamma}^\nu], \quad \tilde{\Gamma}^{\mu 0} = 2i\tilde{\Gamma}^\mu, \quad \tilde{\Gamma}^{0\mu} = -2i\tilde{\Gamma}^\mu \quad (\mu, \nu \neq 0) \tag{A.9}$$

We note that the triality automorphism is implemented by taking an antisymmetric matrix Ω_{IJ}^V in the vector representation to matrices Ω_{AB}^S or $\Omega_{A'B'}^C$ in the spinor representation via

$$\Omega_{\mu\nu}^V \Gamma_{AB}^{\mu\nu} = \Omega_{AB}^S, \quad \Omega_{\mu\nu}^V \tilde{\Gamma}_{A'B'}^{\mu\nu} = \Omega_{A'B'}^C, \tag{A.10}$$

B. Nilpotent orbits in orthogonal groups

In this appendix, we briefly review some general facts about nilpotent co-adjoint orbits, before restricting to orthogonal groups. The proofs of all these results can be found in [29].

Complex nilpotent orbits in G are classified by conjugacy classes of homomorphisms $\mathfrak{su}(2) \to \mathfrak{g}$, i.e. triplets e, f, h of elements in the Lie algebra $\mathfrak{g}$ of G satisfying the SU(2) algebra, $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Under the adjoint action of this SU(2), $\mathfrak{g}$ decomposes into a sum of finite-dimensional representations. $\mathfrak{g}$ may be further decomposed

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[12]We keep the same symbol Γ to avoid unnecessary extra notation.

as a sum of eigenspaces of the Cartan generator h, $\mathfrak{g} = \sum_{i=-i_0 \ldots i_0} \mathfrak{g}_i$. The complex nilpotent orbit is isomorphic to $P \backslash G$, where P is obtained by exponentiating $\mathfrak{p} = \sum_{i \leq 0} \mathfrak{g}_i$. The real dimension of the nilpotent orbit is given by $\dim \mathfrak{g} - \dim \mathfrak{g}_0 - \dim \mathfrak{g}_1 = 2(\dim \mathfrak{g} - \dim \mathfrak{p})$. The set of all nilpotent orbits admits a partial ordering, the closure ordering, whereby $e < e'$ if e' lies in the closure of the nilpotent orbit through e'. All nilpotent orbits of a given group G can be displayed in a Hasse-type diagram, with vertically increasing dimensions and links corresponding to the closure ordering.

For $G = GL(N, \mathbb{C})$, complex nilpotent orbits are in one-to-one correspondence with partitions of N, i.e. Young tableaux with N boxes. The partition corresponds to the Jordan normal form of the nilpotent element e, or to the dimensions of the representations appearing in the decomposition of (g) under SU(2). For $G = SO(N, \mathbb{C})$, complex nilpotent orbits are in one-to-one correspondence with Young tableaux with N boxes such that lines of even length always occur in pairs. When N is even, "very even" partitions, corresponding to configurations with only rows of even length, are an exception to this rule, as they label two distinct orbits. For $G = GL(N, \mathbb{C})$ or $G = SO(N, \mathbb{C})$, the closure ordering $e \leq e'$ holds whenever for all $p = 1 \ldots N$, the number of boxes in the first p columns of the Young tableau associated to e is less than the number of boxes in the first p columns of the Young tableau associated to e' (see [30] for a physical realization of this ordering).

In table 1, we list the complex nilpotent orbits of $G = SO(n_v + 10, \mathbb{C})$ whose dimension scales as $k n_v$ with $k \leq 8$ when $n_v \to \infty$;[13] their closure relations are displayed in the Hasse diagram in figure (1). The table reveals two complex nilpotent orbits whose real dimension equals the real dimension $8 n_v + 28$ of the twistor space Z. The nilpotent orbit $(5, 2^4, 1^{n_v-3})$ corresponds to a weight decomposition ranging from $i = -6$ to $i = 6$ and bears no relation with Z. In contrast, the nilpotent orbit $(3^4, 1^{n_v-2})$ gives rise to the same 5-grading as in (3.17),

$$\mathfrak{g} = 6|_{-4} \oplus 4(n_v + 2)|_{-2} \oplus \frac{1}{2}(n_v^2 + 3n_v + 34)|_0 \oplus 4(n_v + 2)|_2 \oplus 6|_4 \qquad (B.1)$$

and so is identical to the twistor space Z (3.2). It may be worthwhile noting that the orbit $(2^4, 1^{n_v+2})$ yields the same grading, but with half the charge; as a result its dimension is smaller by $4(n_v + 2)$. On the other hand, the orbit $(3^2, 1^{n_v+4})$, of real dimension $4n_v + 26$, gives the same 5-grading as (2.12),

$$\mathfrak{g} = 1|_{-4} + (2n_v + 12)|_{-2} + \frac{1}{2}(n_v^2 + 11n_v + 38)|_0 + (2n_v + 12)|_2 \oplus 1|_4 \qquad (B.2)$$

which is adapted to the complex structure on the twistor space of the quaternionic-Kähler manifold $SO(4, n_v + 6)/SO(4)SO(n_v + 6)$. Again, the orbit $(2^2, 1^{n_v+6})$ gives the same grading but with half the charge. Finally, the orbit $(3, 1^{n_v+7})$ of real dimension $2(n_v + 8)$ gives a three-grading

$$\mathfrak{g} = (n_v + 8)|_{-2} \oplus \frac{1}{2}(n_v^2 + 15n_v + 58)|_0 \oplus (n_v + 8)|_2 \qquad (B.3)$$

adapted to the complex structure on $SO(2, n_v + 8)/SO(2) \times SO(n_v + 8)$.
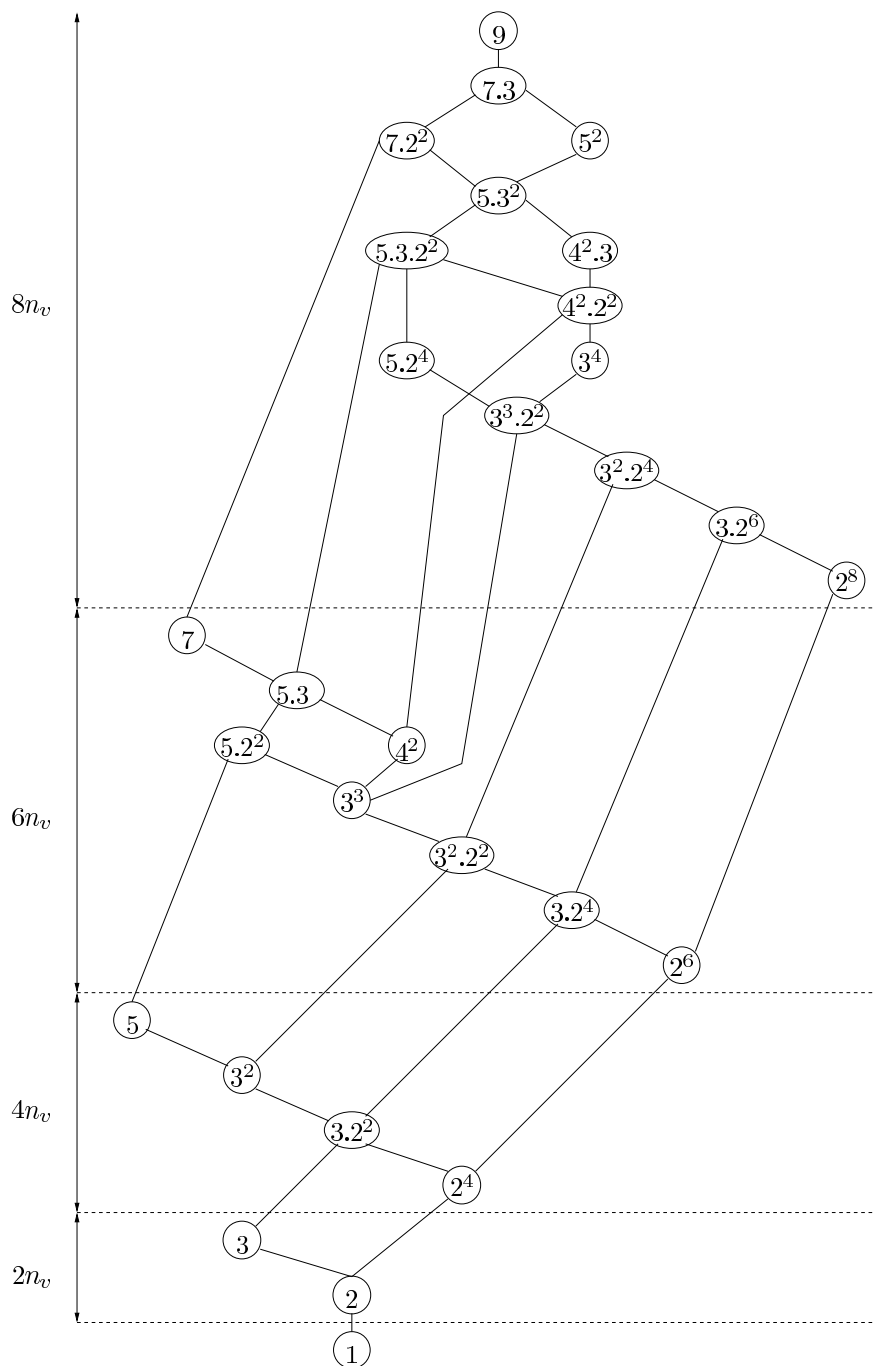
Figure 1: Hasse diagram for the nilpotent orbits in table 1.

We now turn to the classification of real nilpotent orbits, which is rather more subtle. For real orthogonal groups $SO(p, q, \mathbb{R})$, nilpotent orbits are classified by Young tableaux with $N = p + q$ boxes as above, with additional assignments of a sign $\pm$ to each box such

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[13]It is easy to see that the numbers of nilpotent orbits whose dimension scales as kn_v is given by the coefficient of q^k in the Taylor expansion of $1/\prod_{n=0}^{\infty}(1 - q^{4n+2})(1 - q^{2n+2})$ around $q = 0$.

$(2^2, 1^{n_v+6})$	$: 2n_v + 14$		
$(3, 1^{n_v+7})$	$: 2n_v + 16$	$(2^8, 1^{n_v-6})$	$: 8n_v + 8$
		$(3, 2^6, 1^{n_v-5})$	$: 8n_v + 16$
$(2^4, 1^{n_v+2})$	$: 4n_v + 20$	$(3^2, 2^4, 1^{n_v-4})$	$: 8n_v + 22$
$(3, 2^2, 1^{n_v+3})$	$: 4n_v + 24$	$(3^3, 2^2, 1^{n_v-3})$	$: 8n_v + 26$
$(3^2, 1^{n_v+4})$	$: 4n_v + 26$	$(3^4, 1^{n_v-2})$	$: 8n_v + 28$
$(5, 1^{n_v+5})$	$: 4n_v + 28$	$(5, 2^4, 1^{n_v-3})$	$: 8n_v + 28$
		$(4^2, 2^2, 1^{n_v-2})$	$: 8n_v + 30$
$(2^6, 1^{n_v-2})$	$: 6n_v + 18$	$(4^2, 3, 1^{n_v-1})$	$: 8n_v + 32$
$(3, 2^4, 1^{n_v-1})$	$: 6n_v + 24$	$(5, 3, 2^2, 1^{n_v-2})$	$: 8n_v + 32$
$(3^2, 2^2, 1^{n_v})$	$: 6n_v + 28$	$(5, 3^2, 1^{n_v-1})$	$: 8n_v + 34$
$(3^3, 1^{n_v+1})$	$: 6n_v + 30$	$(7, 2^2, 1^{n_v-1})$	$: 8n_v + 36$
$(4^2, 1^{n_v+2})$	$: 6n_v + 32$	$(5^2, 1^{n_v})$	$: 8n_v + 36$
$(5, 2^2, 1^{n_v+1})$	$: 6n_v + 32$	$(7, 3, 1^{n_v})$	$: 8n_v + 38$
$(5, 3, 1^{n_v+2})$	$: 6n_v + 34$	$(9, 1^{n_v+1})$	$: 8n_v + 40$
$(7, 1^{n_v+3})$	$: 6n_v + 36$		

Table 1: Complex nilpotent orbits in $SO(n_v + 10, \mathbb{C})$ with dimension $\mathcal{O}(kn_v)$ with $k \leq 8$.

that signs alternate along lines, rows of even length start with $+$, and the total number of (plus,minus) signs is (p, q). A given signed Young tableau may corresponds to 4 different orbits when all rows have even length, 2 different orbits when all rows with odd length have an even number of $+$, 2 different orbits when all rows with odd length have an even number of $-$, and a unique orbit in other cases [29]. For example, $SO(8, 2, \mathbb{R})$ admits 7 non-zero nilpotent real orbits, corresponding to the partitions

$$(2^2, 1^6), \quad (3, 1^7)_{I,II,III}, \quad (3^2, 1^4), \quad (5, 1^5)_{I,II} . \tag{B.4}$$

of dimension 14, 16, 26 and 28 and nilpotency degree 2,3,3,5, respectively. In particular, there are 4 inequivalent nilpotent orbits of degree 3, none of whose dimension agrees with the dimension Z (the orbits $(5, 1^5)_{I,II}$ do happen to have dimension 28, but are related to a 13-th grading, as indicated above). This is an artifact of this low-rank case, since the nilpotent orbit $(3^4, 1^{n_v-2})$ does appear in the list of real nilpotent orbits of $SO(8, 2 + n_v, \mathbb{R})$ for $n_v \geq 2$. Choosing the sign configuration $[(+ - +)^4, (-)^{n_v-2}]$, all rows have odd length and carry an odd number of minuses, so this configuration appears in two varieties, related by an outer automorphism of $SO(8, n_v + 2)$.

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